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Iakov Gurevich

Research adviser:

Rzayev Ruslan Aliyevich

ENUMERATION OF CORRESPONDENCES BETWEEN FINITE SETS

This work is a combinatorial study of classical algebraic objects – correspondences (binary relations) on finite sets. The main result is the enumeration of difunctional correspondences. Three new combinatorial sequences have been found and registered in OEIS.

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1. Research objective

1.1. Combinatorics

Combinatorics is one of the oldest sections of mathematics. The classic problem of combinatorial science: "How many ways are there to extract m elements from N possible ones" is mentioned in the sutras of ancient India (since about 4th century B.C.). Indian mathematicians discovered binomial coefficients and already in the 2nd century B.C. knew that the sum of all binomial coefficients of degree n is 2n (Wikipedia).
The main problem of enumerative combinatorics is to enumerate (calculate) different objects with given properties. Formally, an enumeration requires the presentation of a series of natural numbers (extended 0), usually infinite, where the i-th term is the desired number for the value of i. In the event of several parameters, the series is formed by a definite rule of table (multidimensional table) traversal. A classical solution involves finding some formula that allows you to calculate a term of this series directly or recurrently, i.e., depending on the previous terms of the series. For some sequences where a formula cannot be constructed, approximate formulas, upper and lower estimates are found. Recently, interest has increased in algorithms and programs that allow you to calculate a certain number of initial terms of this series in the case when the formula cannot be found. At that, the labor intensity of combinatorial problems is extremely high, and only a very limited number of such terms can be counted by brute force. Most often, such algorithms generate all objects with specified properties. It is necessary to prove that a) the algorithm will generate all objects; b) no unnecessary objects will be generated; c) no objects will be generated more than one time. Besides, there are specific difficulties in such programs, for example, (G. Brinkmann, 2002).
The article (Wikipedia) in the combinatorial problem list with the number 2 contains the problem: "How many functions F exist from an n-element set to an m-element set that satisfy the given constraints?" It seemed to us that it is possible to slightly expand the problem.

1.2. Basic concepts

In algebra classics, the binary relation $\boldsymbol{R}$ over a pair of sets $\boldsymbol{A}$, $\boldsymbol{B}$ is called any subset of a Cartesian product of these sets $\boldsymbol{R} \subseteq \boldsymbol{A} \times \boldsymbol{B}$ (A. Maltsev, 1970). In works on set theory, the set $\boldsymbol{R} \subseteq \boldsymbol{A} \times \boldsymbol{B}$ is called "graph", and the triple $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$ is a correspondence (N. Bourbaki, 1965, Yu. Shikhanovich, 1965, and I. Vinogradov, 1985). A special case is singled out when A=B. In this case, the correspondence is called the relation over set. It is understood as the pair $< \boldsymbol{R}, \boldsymbol{A} >$.

The following properties are defined for correspondences: functionality, total/complete definability, injectivity, and surjectivity (Yu. Shikhanovich, 1965 and I. Vinogradov, 1985). The property of difunctionality is less frequently found in the literature. This concept was introduced by J. Riguet in 1948 (translated into Russian in 1963). This definition can also be found in (A. Maltsev, 1970; I. Vinogradov, 1985; and Wikipedia).

Correspondences that have a feature of functionality are called functions. In some sources, only total functional correspondences are considered functions. In any case, a function is a special case of correspondence. We haven't been able to find a systematic enumeration of correspondences with all possible property sets, as it is done for graphs (E. Palmer, 1977) or relations over set (G. Pfeiffer, 2004).

1.3. The objective of this work is to enumerate the correspondences between finite n-element and m-element sets with all possible combinations of properties.

Let us arrange the result in the form of a table:

	D	F	C	I	S	Formula	Note	OEIS
1								
2					+			
3				+				
...	...	...	...	...	...			
32	+	+	+	+	+			

The "+" sign notes the corresponding property in the correspondence. The strings are numbered in such a way that it is easy to understand by number what properties should have the correspondence from this string. For example, in string 12, the number of the string shall be decreased by 1 and expand in powers of twos: $12-1=11=2^3+2^1+2^0$. 0 is the surjectivity, 1 is the injectivity, 2 is the complete definability, 3 is the functionality, and 4 is the difunctionality.

1.3.1. To achieve this goal, it is necessary to solve $32=2^5$ combinatorial problems (to fill the strings of the table) of different complexity (except for the count of strings representing problems).

1.3.2. The first problem will be considered solved if a formula is found to enumerate the correspondences having the set of properties specified in the i-th string. Recurring formulas are allowed.

1.3.3. To analyze the result, it is also planned to search the sequences found in OEIS (N.J.A. Sloane). To do this, we plan to develop a program that generates an integer sequence for each string using the obtained formulas by the table traversal using the "antidiagonal" way as it is customary in OEIS.

1.3.4. Besides, for additional control, it was decided to develop an application for calculating the correspondences by brute force.

1.3.5. The most significant result is the finding of any new sequence and its registration in OEIS. In this case, it is also necessary to understand the rules of the application registration in OEIS.

1.4. OEIS

The On-Line Encyclopedia of Integer Sequences (OEIS) is the most comprehensive reference for integer sequences. It contains more than 250 thousand numeric integer sequences collected in the form of articles. Each article contains the initial numbers of the series, a brief description of the mathematical meaning, and a formula if it has been found. The formation of this encyclopedia was started in 1965 by Neil Sloan, a combinatorial specialist. It was published in printed form at that time. From the beginning, the OEIS contained well-known combinatorial series such as Stirling numbers or Bell numbers.

Currently, the OEIS is a public library available on the Internet. The search tools allow you to find the relevant article(s) by several sequence members. The OEIS has fundamentally changed the ability to quickly assess the novelty of the found sequence, primarily in the combinatorics. When finding a new sequence, everyone can register it and, having passed a multi-level check by the editors, and publish it.

1.5. Related problems

The impetus for this work was an article (G. Pfeiffer, 2004) in the Journal of Integer Sequences, which tries to enumerate relations over set with all kinds of property bags.

For relations over set < ***R***, ***A*** >, where $R \subseteq A \times A$, the following properties are defined: symmetry, antisymmetry, reflexivity, antireflexivity, and transitivity. Such properties, like Euclidicity and completeness, are used less frequently. Relations are very tightly bound to oriented graphs, the study of which occupies a large place in discrete mathematics. Probably, this is why more attention is paid to the problem of enumerating relations. However, the problems of enumerating a whole range of transitive relations, including for the purposes of arbitrary transitive and order relations, have not been solved yet. The paper (G. Pfeiffer, 2004) shows that all unsolved problems are reduced to the enumeration problem of partial order relations. The paper (G. Brinkmann, 2002) presents the results of generating partial order relations up to n=18. Interestingly, this took about 30 years of machine time. The computations were performed on more than 200 computers and revealed some specific problems.

2. Methodology

Combinatorics is a well-developed section of mathematics, and it contains a large number of ways to calculate objects. In the modern approach, combinatorial problems are usually described as problems of calculating the number of elements of finite sets with definite properties (N. Vilenkin, 1976; Yu. Shikhanovich, 1965; and K. Rybnikov, 1985). Finite sets assume some simplifications, which we will use. To describe the basic combinatorial methods, let us introduce two concepts. The potency or cardinality measure (cardinal) of finite set A ($|A|$) is the number of its elements. Following the book (N. Vilenkin, 1976) about the set A, $|A|$=n, we will say “n-set” because, in most cases, the nature of the elements of sets is not essential for us. The power set A ($\mathcal{P}(A)$) is the set of all its subsets. Basic methods of combinatorics are the following.

2.1. Equivalence of sets determination

2.1.1. If a bijective (unambiguous) correspondence can be determined between two finite sets, then these sets contain the same number of elements. This statement can be easily proved for finite sets by induction. In general, the term "equipotent" sets is based on the determination of the bijection between them.
2.1.2. If it is possible to determine bijection between sets, it is sufficient to enumerate any of them because they are equipotent. This method allows you to reduce some problems to

the enumeration of objects that are more studied (often with already known results) (2.6) or with a more obvious way to achieve a result.

2.2. Method of mathematical induction

If P(1) is true and from the truth of P(n) follows the truth of P(n+1), then P(k) is true for all natural k (N. Vilenkin, 1976).

2.3. Method of addition

Let A_1, A_2 be the finite sets and $A_1 \cap A_2 = \emptyset,$ then $|A_1 \cup A_2| = |A_1| + |A_2|$.

More complicated form. Let there be n pairwise disjoint finite sets $A_1, A_2, \dots, A_n$

$$(\forall\, i, j \in \{1,2,\dots,n\}\ \ i \neq j\ \ A_i \cap A_j = \emptyset) \implies \left|\bigcup_{k=1}^{n} A_k\right| = \sum_{k=1}^{n} |A_k|$$

(N. Vilenkin, 1976).

2.4. Multiplication method

Let A_1, A_2 be the finite sets, then $|A_1 \times A_2| = |A_1| * |A_2|$

Or for n finite sets $A_1, A_2, \dots, A_n$ it is true

$$|A_1 \times A_2 \dots \times A_n| = \prod_{i=1}^{n} |A_i|$$

Sometimes it is more convenient to reformulate the method. If the 1st element of a tuple of length **n** can be selected by $\mathbf{k_1}$ ways, the 2nd - by $\mathbf{k_2}$ ways, i-th - by $\mathbf{k_i}$ ways, then the tuple can be built by $\prod_{i=1}^{n} k_i$ ways (N. Vilenkin, 1976).

2.5. Inclusion-exclusion method

2.5.1. It is a generalization of the method of addition for the case of intersecting sets (N. Vilenkin, 1976). (A_1 A_2 are finite sets).

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

2.5.2. For the intersection, it is possible to use the formula translation

$$|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2|.$$

2.5.3. Or in the general case. Let there be n finite sets $A_1, A_2, \dots, A_n$

$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{k=1}^{n} |A_k| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_j| + \dots + (-1)^{m-1} \left|\bigcap_{i=1}^{m} A_i\right|$$

2.6. Reducing problems to classical problems

Let us present the most well-known of them.

2.6.1. The power set potency of the finite n-set is equal to 2^n .

$$|\boldsymbol{\mathcal{P}}(\boldsymbol{A})| = \mathbf{2}^{|\boldsymbol{A}|}$$

From this, it follows that the number of strings in our table is 2^5. Really, we have a set of 5 different properties. Any subset is the string in our table.

2.6.2. The combination is k-subset of the n-set. The number of combinations is determined by the formula:

$$\boldsymbol{C_n^k} = \frac{\boldsymbol{n}!}{\boldsymbol{k}!\,(\boldsymbol{n}-\boldsymbol{k})!}$$

2.6.3. The placement is tuples of length *k*, where all components are different and represent elements of the n-set

$$\boldsymbol{A_n^k} = \frac{\boldsymbol{n}!}{(\boldsymbol{n}-\boldsymbol{k})!} = \boldsymbol{C_n^k} * \boldsymbol{k}!$$

2.6.4. The placement of length n over the n-set is called a permutation. The number of permutations is **n!**

2.6.5. The placement with repetitions is a tuple of length k, where all components are elements of the n-set

$$\overline{\boldsymbol{A}}_{\boldsymbol{n}}^{\boldsymbol{k}} = \boldsymbol{n}^{\boldsymbol{k}}$$

2.6.6. The number of separations of the n-set into k non-intersecting subsets is known as Stirling numbers of the 2nd kind. The recurrence formula is easy to prove:

$$\boldsymbol{S_{n+1}^k} = \boldsymbol{k} * \boldsymbol{S_n^k} + \boldsymbol{S_n^{k-1}}$$

2.7. Stirling transform

We will also use the less common method – the Stirling transform (M. Bernstein). The Stirling transform of the sequence $A = (a_1, a_2, \dots, a_{n\dots})$ is called the sequence

$\boldsymbol{B} = (\boldsymbol{b_1}, \boldsymbol{b_2}, \dots, \boldsymbol{b_n}\ \dots)$, where $\boldsymbol{b_n} = \sum_{\boldsymbol{k}=\boldsymbol{1}}^{\boldsymbol{n}} \boldsymbol{S_n^k} * \boldsymbol{a_k}$

where $\boldsymbol{S_n^k}$ is the Stirling numbers of the 2nd kind.

The combinatorial meaning of this transformation is as follows. If a_k is the number of some combinatorial objects on the separation of the set A into k classes, and different separations provide different combinatorial objects, there are $\sum_{k=1}^{n} S_n^k * a_k$ such combinatorial objects on the n-set. The simplest example of using the Stirling transform is to calculate the equivalence relations over set (Bell numbers). Each separation of a set corresponds to the equally one equivalence, and, in this case, the series to which the Stirling transform is applied is just a sequence of units:

$$B_n = \sum_{k=1}^{n} S_n^k * 1$$

3. Basic concepts

3.1. Terminology and notations

The mathematical literature on algebra and combinatorics concentrates mainly on mappings and bijections. Arbitrary correspondences are studied to a much lesser extent. It has led to ambiguity even though in terminology. For example, in (Ju. Schreider, 1971, p. 21), no difference is made between functions and mappings, and in (A. Maltsev, 1970, p. 33), the mapping is considered the basic concept, and so, arbitrary correspondences are called partial multimappings and functions are called partial mappings. In (J. Riguet, 1963, p. 146), not total functions are called "quasi-functions." Following this terminology, for example, the function y=1/x should be considered a "quasi-function." In the English sources (C.Brink, 1997), we also find the term "partial function." We will mainly follow the terminology provided in works (Yu. Shikhanovich, 1965; I. Vinogradov, 1985; and N. Bourbaki, 1965). To avoid ambiguity, let us present the fundamental definitions.

3.1.1. Triple of sets $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$, where $\boldsymbol{R} \subseteq \boldsymbol{A} \times \boldsymbol{B}$ is called the correspondence between sets $\boldsymbol{A}$ and $\boldsymbol{B}$. $\boldsymbol{R}$ is the graph of the correspondence, $\boldsymbol{A}$ is the *domain of departure*, and $\boldsymbol{B}$ is the *domain of arrival*..
3.1.2. $\boldsymbol{X} \subseteq \boldsymbol{A}$ is the *range of definition*, if $(\boldsymbol{x} \in \boldsymbol{X}) \Leftrightarrow (\exists < x, y > \in R)$.
3.1.3. $\boldsymbol{Y} \subseteq \boldsymbol{B}$ is the range of values, if $(\boldsymbol{y} \in \boldsymbol{Y}) \Leftrightarrow (\exists < x, y > \in R)$.
3.1.4. The correspondence is *functional*, if $(< x, y > \in R) \&$
$(< x, z > \in R) \Rightarrow (\mathbf{y} = \mathbf{z})$.
3.1.5. The correspondence is *injective*, if $(< x, y > \in R) \& (< z, \boldsymbol{y} > \in R) \Rightarrow (\mathbf{x} = \mathbf{z})$.
3.1.6. The correspondence is total if the domain of arrival coincide with the range of definition $(\forall\, \boldsymbol{a} \in \boldsymbol{A})\ (\exists\, \boldsymbol{b} \in \boldsymbol{B})\ [< a, b > \in R\ ]$.
3.1.7. The correspondence is *surjective*, if the domain of arrival coincide with the range of values $(\forall\, \boldsymbol{b} \in \boldsymbol{A})\ (\exists\, \boldsymbol{a} \in \boldsymbol{B})\ [< a, b > \in R]$.
3.1.8. The correspondences that have the property of functionality, we will call *functions*.
3.1.9. Mappings are total functions.
3.1.10. *Bijection* or unambiguous correspondence is simultaneously functional, total, injective, and surjective correspondence.
3.1.11. *Injective mapping* is the injective function.
3.1.12. Let $\boldsymbol{R} \subseteq \boldsymbol{A} \times \boldsymbol{B}$ be the graph. The set $\{\boldsymbol{b} \in \boldsymbol{B}: \exists \boldsymbol{a} \in \boldsymbol{D}\ [< a, b > \in R]\}$ is called the image of set D (D⊆A) (R(D)). For short, we will also write R(x) instead of R({x}} and say the "image of an element."
In the future we will keep the notations $\boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{X}, \boldsymbol{Y}$ for the corresponding sets. In addition, we believe that $|A| = n, |B| = m$

In cases where there will be no confusion, we will denote properties of difunctionality, functionality, complete definability, injectivity, and surjectivity by capital letters D, F, C, I, and S. We will, where convenient, write xRy, which means <x,y> ∈ $\boldsymbol{R}$.

We will denote by R**** a correspondence graph with properties specified in the top index (for example, R^{FI} – the graph with functionality and injectivity). We will denote by V**** a correspondence that simultaneously has properties specified in the top index.

We will denote a set of matches between n-set and m-set that satisfy the conditions of the i-th string by W_i (n,m). We will also denote a set of correspondences that simultaneously has properties specified in the top index by W^{****}(m,n) (e.g. W^{FC} is functionality and complete definability).

To calculate the number of correspondences, we will number all elements of the domain of departure and domain of arrival by consecutive natural numbers. Obviously, for the enumeration, it is possible to replace the elements themselves with their indexes.

Let us introduce, following (Yu. Shikhanovich, 1965), "the language of arrows." Let us denote the elements of the set of the domain of departure by dots. On the right, let us depict the elements of the set of the domain of arrival with dots. If there is a pair <x,y> in the graph, we will connect the points x and y with an arrow. Then we can give the following definitions of the basic properties.

3.1.13. Let us define $\boldsymbol{R^{-1}}$ which is contrary to the graph $\boldsymbol{R} \subseteq \boldsymbol{A} \times \boldsymbol{B}$

$$\boldsymbol{R^{-1}} \subseteq \boldsymbol{B} \times \boldsymbol{A}; < b, a > \in \boldsymbol{R^{-1}} \Leftrightarrow < a, b > \in \boldsymbol{R} \text{ .}$$

3.1.14. Let us define the operation of composition on the graphs

$$\boldsymbol{P} \subseteq \boldsymbol{A} \times \boldsymbol{Z}, \boldsymbol{Q} \subseteq \boldsymbol{Z} \times \boldsymbol{B}; \;\; \boldsymbol{R} \subseteq \boldsymbol{A} \times \boldsymbol{B};$$

$$\boldsymbol{R} = \boldsymbol{P} \circ \boldsymbol{Q}: \; < a, b > \in \boldsymbol{R} \Leftrightarrow \exists \boldsymbol{z} \in Z \; [< a, z > \in \boldsymbol{P} \; \& \; < z, b > \in \boldsymbol{Q}]$$

3.1.15. The correspondence is called contrary to the given one, if the domain of its departure is the domain of arrival of the given one, the domain of arrival is the domain of departure, and the graph is a reciprocal graph to the given correspondence.

3.1.16. The correspondence W=<R,A,B> is called the composition of correspondences V_1=<R_1,A_1,B_1> and V_2=<R_2,A_2,B_2>, if $A=A_1$; $B=B_2$; $R=R_1 \circ R_2$.

3.1.17. The correspondence whose graph satisfies the ratio below is called a difunctional one (J. Riguet, 1963)

$$\boldsymbol{R} \circ \boldsymbol{R^{-1}} \circ \boldsymbol{R} = \boldsymbol{R}$$

3.2. Properties of correspondences and potency of images

3.2.1. Lemma. The correspondence is functional if and only if (∀a∈A) |R(a)|≤1.

Let us make direct proof by contradiction. Let the functional correspondence be given <R, A, B> and ∃a∈A: |R(a)|>1. Then there is b1≠ b2, such that a R b1 & a R b2. The property of functionality is violated. The contradiction is obtained.

Let us make proof by contradiction, i.e. (∀a∈A) |R(a)|≤1. If follows from the first condition that if ∃a∈A A, there exists $b_1 \neq b_2$ such that a R b_1 & a R b_2, therefore $b_1=b_2$ (because otherwise |R(a)|>2). And this is the definition of functionality.

3.2.2. Lemma. The correspondence is total if and only if (∀a∈A) |R(a)|≥1.

3.2.3. Lemma. The correspondence is the mapping if and only if (∀a∈A) |R(a)|=1.

3.2.4. Lemma. The correspondence is injective if and only if (∀b∈B) $|R^{-1}(b)|\leq 1$.

3.2.5. Lemma. The correspondence is surjective if and only if $(\forall b \in B)\ |R^{-1}(b)| \geq 1$.
3.2.6. Lemma. The correspondence is the bijection if and only if $(\forall a \in A)\ (\forall b \in B)\ (|R(a)|=1$ $|R^{-1}(b)|=1)$.

3.3. Theorem (on conservation of bijection)
Let the correspondence $<R,A,B>$ be a bijection. Let $A_1 \subseteq A$, $B_1 \subseteq B$ be such that $R(A_1) \subseteq B_1$ & $R^{-1}(B_1) \subseteq A_1$. Then the correspondence $< R \cap (A_1 \times B_1), A_1, B_1>$ is the bijection.
The theorem is almost obvious, but, nevertheless, we will prove it rigorously.
3.3.1. Let there be $(\forall a \in A_1) \& (aRb) \Rightarrow (b \in B_1) \Rightarrow <a,b> \in A_1 \times B_1 \Rightarrow <a,b> \in (R \cap (A_1 \times B_1)). \Rightarrow |\ R \cap (A_1 \times B_1)\ (a)|=1$.
3.3.2. Let there be $(\forall b \in B_1) \& (aRb) \Rightarrow (a \in A_1) \Rightarrow <a,b> \in A_1 \times B_1 \Rightarrow <a,b> \in (R \cap (A_1 \times B_1)). \Rightarrow |\ (R \cap (A_1 \times B_1))^{-1}\ (a)|=1$.
The theorem is proved.

3.4. Separations of set

3.4.1. We used the concept of separation to define Stirling numbers. Let us define this concept, according to (Yu. Shikhanovich, 1965). The separation of a nonempty set M is called such a set (system) of sets (classes) $\mathbb{M} = \{\boldsymbol{\mathcal{M}}_1, \boldsymbol{\mathcal{M}}_2, \dots, \boldsymbol{\mathcal{M}}_k\}$ that the following is executed:

1) $(\forall\ \boldsymbol{\mathcal{M}} \in \mathbb{M})\ [\boldsymbol{\mathcal{M}} \subseteq \boldsymbol{M}]$
2) $(\forall\ \boldsymbol{\mathcal{M}} \in \mathbb{M})\ [\boldsymbol{\mathcal{M}} \neq \emptyset]$
3) $(\forall\ \boldsymbol{\mathcal{M}}_1, \boldsymbol{\mathcal{M}}_2 \in \mathbb{M})[\ \boldsymbol{\mathcal{M}}_1 \neq \boldsymbol{\mathcal{M}}_2 \Rightarrow \boldsymbol{\mathcal{M}}_1 \cap \boldsymbol{\mathcal{M}}_2 = \emptyset]$
4) $\mathrm{M} = \bigcup_{\boldsymbol{\mathcal{M}} \in \mathbb{M}} \boldsymbol{\mathcal{M}}$

The following properties (3.4.2-3.4.5) of the separations are obvious and we will use them in the future.

3.4.2. Lemma. $(\forall \mathrm{m} \in \mathrm{M})(\exists!\ \boldsymbol{\mathcal{M}} \in \mathbb{M})\ [\mathrm{m} \in \boldsymbol{\mathcal{M}}]$. That $\boldsymbol{\mathcal{M}}$ exists follows from the axiom 4), and the uniqueness – from the axiom 3).
3.4.3. Lemma. $(\forall \boldsymbol{\mathcal{M}} \in \mathbb{M})(\exists \mathrm{m} \in \mathrm{M})\ [\mathrm{m} \in \boldsymbol{\mathcal{M}}]$. It follows from the axioms 1) and 2).
3.4.4. Lemma. $(\forall\ \boldsymbol{\mathcal{M}}_1, \boldsymbol{\mathcal{M}}_2 \in \mathbb{M})[\ \boldsymbol{\mathcal{M}}_1 \cap \boldsymbol{\mathcal{M}}_2 \neq \emptyset \Rightarrow \boldsymbol{\mathcal{M}}_1 = \boldsymbol{\mathcal{M}}_2]$. It follows directly from the axiom 3).
3.4.5. Lemma. The maximal cardinal of the separation of the set M is equal to |M|, the minimal cardinal is equal to 1. Let $\exists \mathbb{M}$: $|\ \mathbb{M}\ | > |\mathrm{M}|$. By the rule of sum $|\bigcup_{\boldsymbol{\mathcal{M}} \in \mathbb{M}} \boldsymbol{\mathcal{M}}| > |\mathrm{M}|$ but, according to axiom 4), $|\bigcup_{\boldsymbol{\mathcal{M}} \in \mathbb{M}} \boldsymbol{\mathcal{M}}| = |\mathrm{M}|$. The contradiction is obtained. Obviously, the minimum cardinal is equal to 1.
3.4.6. The pair Z=<R,M>, where $R \subseteq M \times M$, is called the relation over set. Actually, the relation is the correspondence with coincident domains of departure and arrival.
The relation <R,M> is called conjugate with the separation $\mathbb{M}$ if

$$(\forall\ a \in M)(\forall b \in M)\{(aRb)\ (\exists \mathcal{M} \in\ \mathbb{M})\ [(a \in \mathcal{M}) \& (b \in \mathcal{M})]\ \}$$

3.4.7. Lemma. The relation conjugate with the separation is the equivalency.

3.4.8. The equivalence relations over set and the separations of this set bijectively define each other. The proof of these lemmata can be found, for example, in (Yu. Shikhanovich, 1965).

3.5. Kronecker symbol

3.5.1. It is convenient to use the Kronecker symbol (equality indicator) to record some formulas:

$$\boldsymbol{\delta}_j^i = \begin{cases} \mathbf{1}, \boldsymbol{i} = \boldsymbol{j} \\ \mathbf{0}, \boldsymbol{i} \neq \boldsymbol{j} \end{cases}, \text{where } \boldsymbol{i}, \boldsymbol{j} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$$

3.5.2. Lemma. Let there be a sequence of numbers $x_1, x_2, \ldots, x_m$. Then the equality is true

$$\sum_{i=1}^{l} \boldsymbol{\delta}_i^k \boldsymbol{x}_i = \boldsymbol{x}_k$$

under any $\boldsymbol{k} \leq \boldsymbol{l} \leq \boldsymbol{m}$.

Actually, all terms in the case of $\boldsymbol{i} \neq \boldsymbol{k}$ are equal to 0. Since i runs up to $\boldsymbol{l} \geq \boldsymbol{k}$, the only non-zero term is $\boldsymbol{\delta}_k^k \boldsymbol{x}_k = \boldsymbol{x}_k$.

4. Enumerating correspondences without considering the property of difunctionality

4.1. Reducing a part of problems to others

Some problems can be reduced to others, thus reducing their quantity.

4.1.1. Lemma. There is evenly one graph opposite of the given one. For each graph, we can build at least one opposite, just by "turning over" all the tuples. It is easy to see that such a graph will be the opposite of the given one. Let us prove by contradiction that the number of graphs is not more than one. Let there be

$$\boldsymbol{R}_1^{-1} \text{ and } \boldsymbol{R}_2^{-1};\ \boldsymbol{R}_1^{-1} \neq \boldsymbol{R}_2^{-1};$$

$\exists < \boldsymbol{y}, \boldsymbol{x} >: < y, x > \in \boldsymbol{R}_1^{-1} \ \& < y, x > \notin \boldsymbol{R}_2^{-1}$.

$< y, x > \in \boldsymbol{R}_1^{-1} \quad < x, \boldsymbol{y} > \in \boldsymbol{R} \Rightarrow < \boldsymbol{y}, \boldsymbol{x} > \in \boldsymbol{R}_2^{-1}$.

The contradiction is obtained.

4.1.2. Lemma. For the given graph, there is evenly one graph opposite of the given one. Again, by "turning over" the tuples, we will present at least one initial graph. Let us prove by contradiction that the number of initial graphs is not more than one. Let there be $\boldsymbol{R}_1$ and $\boldsymbol{R}_2 : \boldsymbol{R}_1 \neq \boldsymbol{R}_2 ;\ \boldsymbol{R}_1^{-1} = \boldsymbol{R}_2^{-1}$

$\exists < \boldsymbol{x}, \boldsymbol{y} >: < \boldsymbol{x}, \boldsymbol{y} > \in \boldsymbol{R}_1 \ \& < \boldsymbol{x}, \boldsymbol{y} > \notin \boldsymbol{R}_2$.

$< x, y > \in \boldsymbol{R}_1 \ \Rightarrow < y, \boldsymbol{x} > \in \boldsymbol{R}_1^{-1} \Rightarrow < y, x > \in \boldsymbol{R}_2^{-1} \Rightarrow < \boldsymbol{x}, \boldsymbol{y} > \in \boldsymbol{R}_2$.

The contradiction is obtained.

4.1.3. Complementation. The correspondence between graphs and their opposite graphs is bijective. It follows from 3.2.6.

4.1.4. Lemma. The correspondence is injective if and only if the correspondence opposite of it is functional. Let there be $R' = (R^I)^{-1}$. Let us prove by contradiction that R' is functional.

$(\exists \boldsymbol{y} \in \boldsymbol{B}, \exists \boldsymbol{x}_1, \boldsymbol{x}_2 \in \boldsymbol{A})\ [\boldsymbol{y}\boldsymbol{R}'\boldsymbol{x}_1 \& \ yR'\boldsymbol{x}_2] \ \Rightarrow \boldsymbol{x}_1\boldsymbol{R}\boldsymbol{y} \ \& \ \boldsymbol{x}_2\boldsymbol{R}\boldsymbol{y}$

The contradiction is obtained.

Let us now prove by contradiction in reverse. So, if R^{-1} is functional, then R is injective. Let us prove by contradiction.

$$(\exists \boldsymbol{x_1}, \boldsymbol{x_2} \in \boldsymbol{A}, \boldsymbol{x_2} \neq \boldsymbol{x_1}, \exists \boldsymbol{y} \in \boldsymbol{B},)\ [\boldsymbol{x_1 R y} \& \boldsymbol{x_2 R y}]\ \Rightarrow \boldsymbol{y R^{-1} x_1}\ \&\ \boldsymbol{y R^{-1} x_2}$$

The contradiction is obtained.

4.1.5. Complementation. $|W^I(n,m)|=|W^F(m,n)|$ or $|W_3(n,m)|=|W_9(m,n)|$.

For reasons given in 4.1.3 and 4.1.4, we obtain bijection between these sets.

4.1.6. Lemma. The correspondence is surjective if and only if the correspondence opposite of it is total.

4.1.7. Complementation. $|W^S(n,m)|=|W^C(m,n)$ injective and surjective if and only if the correspondence opposite of it is functional and total.

4.1.8. Complementation. $|W^{IS}(n,m)|=|W^{FC}(m,n)|$ or $|W_{13}(n,m)|=|W_4(m,n)|$.

4.1.9. Lemma. The correspondence is total and injective if and only if the correspondence opposite of it is functional and surjective.

4.1.10. Complementation. $|W^{CI}(n,m)|=|W^{FS}(m,n)|$ or $|W_{10}(n,m)|=|W_7(m,n)|$.

4.1.11. Lemma. The correspondence is simultaneously surjective, injective, and total if and only if the correspondence opposite of it is simultaneously total, functional, and surjective.

4.1.12. Complementation. $|W^{CIS}(n,m)|=|W^{CS}(m,n)|$ or $|W_8(n,m)|=|W_{14}(m,n)|$.

4.1.13. Lemma. The correspondence is simultaneously surjective, injective and functional if and only if the correspondence opposite of it is simultaneously total, functional, and inrjective.

4.1.14. Complementation. $|W^{FSI}(n,m)|=|W^{FCI}(m,n)|$, $|W_{12}(n,m)|=|W_{15}(m,n)|$

Lemmata and complementations 4.1.6-4.1.14 are proved similarly as 4.1.4 and 4.1.5.

Thus, strings 2, 3, 4, 7, 8, and 12 are not independent problems and can be reduced to others.

4.2. Problems equivalent to classical ones

Some classical combinatorial problems have many different formulations. The modern formulation, in terms of set theory, was already given above. Previously, formulations about how to place balls into boxes with various additional conditions were popular. Part of the problems to enumerate the correspondences, in fact, is another formulation of the classical problems.

4.2.1. *Correspondences without restrictions* $(\boldsymbol{L_1})$

Theorem. $|W|= 2^{m*n}$.

By multiplication rule (2.4), it is true $|A\times B|=m*n$. Since any subset can be a correspondence graph, the number of arbitrary correspondences is equal (2.6.1) to

$$|\boldsymbol{\mathcal{P}}(\boldsymbol{A}\times\boldsymbol{B})|=2^{|\boldsymbol{A}\times\boldsymbol{B}|}=\boldsymbol{2^{m*n}}$$

4.2.2. *Mappings (* $\boldsymbol{L_{13}}$ ***)***

4.2.2.1. Lemma. $|R^{FC}|=n$.

Really, $(\forall\ a\in A)\ |R(a)|=1$, since *F* requires no more than one, and *T* requires no less than one.

4.2.2.2. Theorem. $|W^{FC}|= m^n$.

Let us build a tuple, each component of which corresponds to a pair from R^{FC}. The length of this tuple is n (4.2.2.1), and the i-th component is $b_i= R^{FC}(\{a_i\})$, $b_i \in B$, $<a_i,b_i> \in R^{FC}$. Obviously, that any graph R^{FC} unambiguously identifies such a tuple and conversely. According to 2.1, it is enough to calculate the number of such tuples. The problem is equivalent to (2.6.5).

4.2.2.3. OEIS.

Functions (L_9).

4.2.2.4. Theorem. $|W^F(n,m)| = (\boldsymbol{m+1})^{\boldsymbol{n}}$.

Similarly to 4.2.2.2, we build a tuple for the graph R^F, where the i-th component contains an image of the i-th element from the domain of departure. In the case when an image is an empty set, let us put at this place a special element – b_0 ($b_0 \notin B$. The problem is equivalent to 2.6.5. Actually, we have determined the bijection between $W^F(n,m)$ and $W^{FC}(n,m+1)$, completed the definition of each graph to total.

4.2.2.5. OEIS.

4.2.3. *Injective mappings (L_{15})*

4.2.3.1. Theorem. $|W^{FCI}(n,m)| = \boldsymbol{A_m^n}$.

Similar to 4.2.2.2, we represent the mapping as a tuple of length n. However, now, due to the surjectivity of the correspondence, tuples may contain only nonrecurrent components. In other words, the problem is reduced to the problem of calculating the placements, according to 2.6.3. Thus, we obtain the formula $\boldsymbol{A_m^n}$.

4.2.3.2. OEIS.

4.2.4. *Bigection* (L_{16})

4.2.4.1. Theorem.$|W^{FCIS}(n,m)| = \delta_n^m * n!$

According to 2.1.1., it is possible to determine the bijection between two finite sets only if their potencies are equal. From there, if n<>m, the number of bijections is 0. If m=n, then the problem is reduced to calculating the number of permutations (2.6.4). The required formula is **n!**

4.2.4.2. OEIS.

4.2.5. *Total correspondences* (L_5)

4.2.5.1. Theorem. $|W^C(n,m)| = (\boldsymbol{2^m - 1})^{\boldsymbol{n}}$.

Again, as in (4.2.2.2), we will build the correspondences as tuples of length n. In i-th place will be $\boldsymbol{R(a_i)}$. Now let us see how many ways are there to build $\boldsymbol{R(a_i)}$. We have only one restriction – the image cannot be empty. In other words, the image is any subset of the domain of arrival set except an empty one. There are $(2^m - 1)$ these images. By the multiplication rule, we obtain the final result $(\boldsymbol{2^m - 1})^{\boldsymbol{n}}$.

4.2.5.2. OEIS A245789.

4.3. Several more complex problems

4.3.1. Surjective mappings (L_{14})

4.3.1.1. Theorem. $|W^{FCS}(n,m)| = \boldsymbol{S_n^m} * \boldsymbol{m}!$.

Each such correspondence generates a separation of the domain of departure A into m non-intersecting classes $\mathbb{A} = \{\mathcal{A}_1, \mathcal{A}_2 \ldots \mathcal{A}_m\}$. In this case, the conjugate equivalence relation is

$x_1 \sim x_2 \Leftrightarrow R(x_1)=R(x_2)$. Really, the equality of images is the equivalence relation within the range of definition (obviously symmetrical, reflexive, and transitive). According to (3.2.3), $(\forall x \in X)[|R(x)|=1]$, consequently, $|\mathbb{A}|=m$.
Let us build the correspondence $<\mathcal{R}, \mathbb{A}, B>$ with the domain of departure $\mathbb{A}=\{\mathcal{A}_1, \mathcal{A}_2 \ldots \mathcal{A}_m\}$ and domain of arrival $B=(\mathcal{A}\ \mathcal{R}\ b) \Leftrightarrow (\forall a \in \mathcal{A})[aRb]$. Every one of these correspondences bijectively defines the initial correspondence. And we only have to count such generated correspondences. This correspondence is a bijection (F, C, S – by construction, injectivity – owing to 3.2.4).
Thus, each separation can generate **m!** bijections (4.2.5.1) with the domain of departure $\mathbb{A}=\{\mathcal{A}_1, \mathcal{A}_2 \ldots \mathcal{A}_m\}$ and domain of arrival B.
The number of such separations is $\boldsymbol{S_n^m}$. By the multiplication rule, we obtain $|W^{FTS}(n,m)|= \boldsymbol{S_n^m} * \boldsymbol{m}!$
4.3.1.2. OEIS

4.3.2. Surjective functions (L_{10})

4.3.2.1. Theorem. $|W^{FS}(n,m)| = \boldsymbol{S_n^{m+1}} * (\boldsymbol{m}+\boldsymbol{1})! + \boldsymbol{S_n^m} * \boldsymbol{m}!$
This case is an extension of case 4.3.1. Let us consider the correspondences separately, where X≠A. Let us define the correspondence V prior to mapping V' in the following way. The domain of arrival is B'=B∪{b0}; b0∉B. For each a∉X, let us assume that aRb0 (V' – by construction of F, C, S). It is easy to see that V and V' define each other bijectively. Thus, we obtain the number of correspondences, where X≠A S_n^(m+1)*(m+1)! By the addition rule, we obtain the final formula:

$$\boldsymbol{S_n^{m+1}} * (\boldsymbol{m}+\boldsymbol{1})! + \boldsymbol{S_n^m} * \boldsymbol{m}!$$

4.3.3. Injective functions (L_{11})

4.3.3.1. Lemma. Let there be V^{FI}=<R,A,B>, where X is the range of definition, Y is the range of values. Then V'=<R,X,Y> is the bijection.
Really, V' retains its functionality and injectivity. However, V' domain of departure coincides with the range of definition, and the domain of arrival coincides with the range of values by construction.
4.3.3.2. Theorem. $|W^{FI}(n,m)|= \sum_{i=0}^{n} \boldsymbol{C_n^i} * \boldsymbol{C_m^i} * \boldsymbol{i}!$
It is easy to see that the correspondence is unambiguously defined by the range of definition, by the range of values, and by the bijective correspondence between them. The number of ways to select the range of definition, such as |X|=i, is equal to $\boldsymbol{C_n^i}$. The number of ways to select the area of values such as |Y|=i is equal to $\boldsymbol{C_m^i}$. The number of bijections between the range of definition and the range of values is equal to i! (4.2.5.1). By the multiplication rule, we obtain $\boldsymbol{C_n^i} * \boldsymbol{C_m^i} * \boldsymbol{i}!$ for the range of definition |X|=i. The range of definition can contain from 0 to n elements. By the addition rule, we obtain the final formula: $|W^{FI}|= \sum_{i=0}^{n} \boldsymbol{C_n^i} * \boldsymbol{C_m^i} * \boldsymbol{i}!$
4.3.4. Total surjective correspondences. This problem appeared to be quite complicated, although the resulting sequence was later found in OEIS.
4.3.4.1. Theorem

$$|\boldsymbol{W}^{\mathrm{CS}}(\boldsymbol{n},\boldsymbol{m})| = (\boldsymbol{2}^{\boldsymbol{m}} - \boldsymbol{1})^{\boldsymbol{n}} - \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{m}-\boldsymbol{1}} \boldsymbol{C}_{\boldsymbol{m}}^{\boldsymbol{i}} * |\boldsymbol{W}^{\mathrm{CS}}(\boldsymbol{n},\boldsymbol{i})|.$$

Let us denote a set of strictly total non-surjective correspondences by $\boldsymbol{W}^{\mathrm{CN}}(\boldsymbol{n},\boldsymbol{m})$. Then we have by the addition rule (2.3) the following

$$\boldsymbol{W}^{\mathrm{C}}(\boldsymbol{n},\boldsymbol{m}) = \boldsymbol{W}^{\mathrm{CS}}(\boldsymbol{n},\boldsymbol{m}) \cup \boldsymbol{W}^{\mathrm{CN}}(\boldsymbol{n},\boldsymbol{m}) \Leftrightarrow$$
$$\left|\boldsymbol{W}^{\mathrm{C}}(\boldsymbol{n},\boldsymbol{m})\right| = \left|\boldsymbol{W}^{\mathrm{CS}}(\boldsymbol{n},\boldsymbol{m})\right| + |\boldsymbol{W}^{\mathrm{CN}}(\boldsymbol{n},\boldsymbol{m})| \Leftrightarrow$$
$$\left|\boldsymbol{W}^{\mathrm{CS}}(\boldsymbol{n},\boldsymbol{m})\right| = \left|\boldsymbol{W}^{\mathrm{C}}(\boldsymbol{n},\boldsymbol{m})\right| - |\boldsymbol{W}^{\mathrm{CN}}(\boldsymbol{n},\boldsymbol{m})|$$

To calculate strictly non-surjective correspondences, let us divide them into (m-1) non-intersecting classes by the cardinal of the range of values. This method ensures that the classes do not intersect. Let us assume that V =<R,A,B>. Y is the domain of arrival. V'=<R,A,Y> is defined bijectively with V. V' is surjective. With the fixed range of values, the number of correspondences for |Y|=i is equal to $|\boldsymbol{W}^{\mathrm{CS}}(\boldsymbol{n},\boldsymbol{i})|$. The number of ways to select a range of values with cardinal i is equal to $\boldsymbol{C}_{\boldsymbol{m}}^{\boldsymbol{i}}$. According to the multiplication rule, the number of strictly non-surjective correspondences with |Y|=i is equal to $\boldsymbol{C}_{\boldsymbol{m}}^{\boldsymbol{i}} * |\boldsymbol{W}^{\mathrm{CS}}(\boldsymbol{n},\boldsymbol{i})|$. By the addition rule, we obtain the following

$$|\boldsymbol{W}^{\mathrm{CN}}(\boldsymbol{n},\boldsymbol{m})| = \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{m}-\boldsymbol{1}} \boldsymbol{C}_{\boldsymbol{m}}^{\boldsymbol{i}} * |\boldsymbol{W}^{\mathrm{CS}}(\boldsymbol{n},\boldsymbol{i})| \Rightarrow$$
$$|\boldsymbol{W}^{\mathrm{CS}}(\boldsymbol{n},\boldsymbol{m})| = (\boldsymbol{2}^{\boldsymbol{m}} - \boldsymbol{1})^{\boldsymbol{n}} - \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{m}-\boldsymbol{1}} \boldsymbol{C}_{\boldsymbol{m}}^{\boldsymbol{i}} * |\boldsymbol{W}^{\mathrm{CS}}(\boldsymbol{n},\boldsymbol{i})|.$$

5. Difunctional correspondences

The most interesting results have been gained by enumerating some difunctional correspondences. Therefore, let us consider them in detail.

5.1. Fundamental definitions and properties of difunctional correspondences

5.1.1. The correspondence is called difunctional if its graph satisfies the ratio (J. Riguet, 1963)

$$\boldsymbol{R} \circ \boldsymbol{R}^{-1} \circ \boldsymbol{R} = \boldsymbol{R};$$

5.1.2. We will, when it is convenient, call difunctional correspondences simply difunctions.
5.1.3. Lemma. $\forall\, \boldsymbol{R} \subseteq \boldsymbol{A} \times \boldsymbol{B}\ \ [\boldsymbol{R} \subseteq \boldsymbol{R} \circ \boldsymbol{R}^{-1} \circ \boldsymbol{R}]$
The proof: $\boldsymbol{xRy} \Rightarrow \boldsymbol{x}(\boldsymbol{R} \circ \boldsymbol{R}^{-1})\boldsymbol{x} \Rightarrow \boldsymbol{x}(\boldsymbol{R} \circ \boldsymbol{R}^{-1} \circ \boldsymbol{R})\boldsymbol{y}$.

5.1.4. Theorem (J. Riguet, 1963). The correspondence $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$ is difunctional if and only if when $\mathbf{R}(\mathbf{a_1}) \cap \mathbf{R}(\mathbf{a_2}) \neq \emptyset \Rightarrow \mathbf{R}(\mathbf{a_1}) = \mathbf{R}(\mathbf{a_2})$.

Let us make direct proof by contradiction. Let us suppose that

$$\boldsymbol{R} = \boldsymbol{R} \circ \boldsymbol{R}^{-1} \circ \boldsymbol{R}; \quad \boldsymbol{R}(\boldsymbol{a_1}) \cap \boldsymbol{R}(\boldsymbol{a_2}) \neq \emptyset \;\&\; R(\boldsymbol{a_1}) \neq \boldsymbol{R}(\boldsymbol{a_2}).$$

$$(\exists \boldsymbol{b_1} \in \boldsymbol{B})\ (\exists\ \boldsymbol{b_2} \in \boldsymbol{B})[\boldsymbol{a_1 R\ b_1} \,\&\, \boldsymbol{a_2 R\ b_1} \,\&\, \boldsymbol{a_2 R\ b_2}\ \&\ < a_1,\ \boldsymbol{b_2} >\notin \boldsymbol{R}] \Rightarrow$$

$$\boldsymbol{a_1 R\ b_1} \,\&\, \boldsymbol{b_1 R^{-1}\ a_2} \ \Rightarrow \boldsymbol{a_1(R\circ R^{-1})\ a_2} \ \Rightarrow \boldsymbol{a_1(R\circ R^{-1}\circ R)\ b_2} \ \Rightarrow \boldsymbol{a_1 R\ b_2}$$

The contradiction is obtained.

Let us make proof by contradiction.
a) Let us consider the case $(\forall\, \boldsymbol{a_1, a_2})\ (\boldsymbol{R(a_1)\cap R(a_2)} = \emptyset)$**.** Then $\boldsymbol{a_1(R\circ R^{-1}\circ R)\ b_1} \ \Rightarrow \boldsymbol{a_1 R\ b_1}$**,** which amounts to $\boldsymbol{R\circ R^{-1}\circ R \subseteq R}$**.** However, owing to 5.1.3 $\boldsymbol{R \subseteq R\circ R^{-1}\circ R}$, and therefore, $\boldsymbol{R = R\circ R^{-1}\circ R}$**.** Which was to be proved.

b) Now, let us consider the case $(\ \exists \boldsymbol{a_1, a_2})\ \ [\boldsymbol{R(a_1)\cap R(a_2)} \neq \emptyset\ \&\ \ \boldsymbol{R(a_1) = R(a_2)}]$**.** Let us make proof by contradiction. Let us suppose that

$$\boldsymbol{a_1(R\circ R^{-1}\circ R)\ b_1} \,\&\, < \boldsymbol{a_1, b_1} >\notin \boldsymbol{R} \Rightarrow \exists \boldsymbol{a_2} \neq \boldsymbol{a_1}\colon\ [\boldsymbol{a_1(R\circ R^{-1})a_2} \,\&\, \boldsymbol{a_2 R\ b_1}\,]$$
$$\Rightarrow (\exists \boldsymbol{a_2}, \exists \boldsymbol{b_2})\colon\ \boldsymbol{a_1 R\ b_2}\ \& \boldsymbol{b_2 R^{-1}\ a_2} \,\&\, \boldsymbol{a_2 R\ b_1}\ \Rightarrow$$

$$(\exists \boldsymbol{a_2}, \exists \boldsymbol{b_2})\colon\ \boldsymbol{a_1 R\ b_2}\ \& \boldsymbol{a_2 R\ b_2} \,\&\, \boldsymbol{a_2 R\ b_1}\,.$$

However, $\boldsymbol{R(a_1) = R(a_2)}$. $\boldsymbol{Owing\ to}$ this, $(\boldsymbol{a_2 R\ b_1}\,) \Rightarrow\ (\boldsymbol{a_1 R\ b_1})$. The contradiction is obtained.

5.1.5. Complementation. The correspondence is difunctional if and only if when

$$\boldsymbol{R^{-1}(b_1)\cap R^{-1}(b_2)} \neq \emptyset\ \ \Rightarrow \boldsymbol{R^{-1}(b_1) = R^{-1}(b_2)}.$$

Let us prove by contradiction into direct side. Let us assume that the correspondence is difunctional and $(\boldsymbol{R^{-1}(b_1)\cap R^{-1}(b_2)} \neq \emptyset)\ \ \& \boldsymbol{R^{-1}(b_1) \neq R^{-1}(b_2)} \Rightarrow$

$$(\,(\exists \boldsymbol{a_1}) : ((\boldsymbol{a_1 R\ b_1})\ \&\ \ (\boldsymbol{a_1 R\ b_2}))\&(\,(\exists \boldsymbol{a_2})\ (\ \boldsymbol{a_2 R\ b_2})\ \&\ \ (\ \boldsymbol{a_2} \notin \boldsymbol{R^{-1}(b_1)}))$$
$$\Rightarrow\quad (\boldsymbol{b_2} \in\ \boldsymbol{R(a_1)\cap R(a_2)})\ \& (\boldsymbol{b_1} \notin\ \boldsymbol{R(a_1)\cap R(a_2)} \Rightarrow$$
$$\boldsymbol{R(a_1)\cap R(a_2)} \neq \emptyset \,\&\, R(\boldsymbol{a_1}) \neq \boldsymbol{R(a_2)}.$$

It contradicts the Riguet's theorem (5.1.4).
Now, let us make proof by contradiction. Let the following be performed

$$\boldsymbol{R^{-1}(b_1)\cap R^{-1}(b_2)} \neq \emptyset\ \ \Rightarrow \boldsymbol{R^{-1}(b_1) = R^{-1}(b_2)}$$

Let us make proof by contradiction. Let us suppose that the correspondence is not difunctional. Then, according to Riguet's theorem (5.1.4)

$$(\,(\exists \boldsymbol{a_1}) : ((\boldsymbol{a_1 R\ b_1})\ \&\ \ (\boldsymbol{a_1 R\ b_2}))\&(\,(\exists \boldsymbol{a_2})\ (\ \boldsymbol{a_2 R\ b_2})\ \&\ \ (\ \boldsymbol{b_1} \notin \boldsymbol{R}\ \ \ (\boldsymbol{a_2}))\ \ \Rightarrow$$
$$(\boldsymbol{a_1} \in \boldsymbol{R^{-1}(b_2)})\& (\boldsymbol{a_1} \in \boldsymbol{R^{-1}(b_1)})\& (\boldsymbol{a_2} \in \boldsymbol{R^{-1}(b_2)}\& (\ \boldsymbol{a_2} \notin \boldsymbol{R^{-1}(b_1)})$$

Thus, $\boldsymbol{a_1} \in \boldsymbol{R^{-1}(b_1)\cap R^{-1}(b_2)} \Rightarrow \boldsymbol{R^{-1}(b_1)\cap R^{-1}(b_2)} \neq \emptyset$ but at the same time $(\boldsymbol{a_2} \in \boldsymbol{R^{-1}(b_2)}\& (\ \boldsymbol{a_2} \notin \boldsymbol{R^{-1}(b_1)} \Rightarrow \boldsymbol{R^{-1}(b_1) \neq R^{-1}(b_2)}$ **.**The contradiction is obtained.

5.2. Reducing some problems to others

5.2.1. Any correspondence having a property of functionality is a difunction.
Let us assume that $R(a_1)\cap R(a_2)\neq\emptyset$. $(|R(a_1)| \leq 1, |R(a_1)| \leq 1) \Rightarrow |R(a_1)\cap R(a_2)|=1 \Rightarrow R(a_1)=R(a_2)$. Owing to 5.1.4, it is proved.

5.2.2. Complementations.
5.2.2.1. $|W^{DF}(n,m)| = |W^{F}(n,m)|$;
5.2.2.2. $|W^{DFC}(n,m)| = |W^{FC}(n,m)|$;
5.2.2.3. $|W^{DFI}(n,m)| = |W^{FI}(n,m)|$;
5.2.2.4. $|W^{DFS}(n,m)| = |W^{FS}(n,m)|$;
5.2.2.5. $|W^{DFCI}(n,m)| = |W^{FCI}(n,m)|$;
5.2.2.6. $|W^{DFIS}(n,m)| = |W^{FIS}(n,m)|$;
5.2.2.7. $|W^{DFCS}(n,m)| = |W^{FCS}(n,m)|$;
5.2.2.8. $|W^{DFCIS}(n,m)| = |W^{FCIS}(n,m)|$;
5.2.3. Any injective correspondence is a difunction.
Really, owing to injectivity:
$R(a_1)\cap R(a_2)\neq\emptyset \Rightarrow a_1 = a_2 \Rightarrow R(a_1)=R(a_2)$. Owing to 5.1.4 , it is proved.
5.2.4. Complementations.
5.2.4.1. $|W^{DI}(n,m)| = |W^{I}(n,m)|$;
5.2.4.2. $|W^{DCI}(n,m)| = |W^{CI}(n,m)|$;
5.2.4.3. $|W^{DIS}(n,m)| = |W^{IS}(n,m)|$;
5.2.4.4. $|W^{DCIS}(n,m)| = |W^{CIS}(n,m)|$;
All complementations 5.2.2 , 5.2.4 are proved uniformly. Let us prove 5.2.2.1.

$$|W^{DF}(n,m)| = |W^{D}(n,m) \cap W^{F}(n,m)| = |W^{F}(n,m)|,$$
$$\text{because } W^{F}(n,m)\subseteq W^{D}(n,m);$$

5.2.5. Thus, strings 19, 20, 23, 24, 25, 26, 27, 28, 29, 30, 31, and 32 are not independent problems and can be reduced to others, already solved ones.

5.3. Generated by difunctions of correspondence

The enumeration of difunctions based on their definition is a problem the way of which solving is not clear. Riguet's theorem gives some idea of how they are "arranged." In the course of further reasoning, we will determine the bijective correspondence between difunctions and a definite class of correspondences, which we will enumerate. As it will become clear later on, these correspondences must have a defined set of properties. We will name their two properties for ease of explanation.
5.3.1. Let us introduce some additional concepts and notations for further explanation.
5.3.2. Let us introduce definitions for the correspondences which have no more than one element that is not included in the range of definition and, similarly, for the range of values.
5.3.3. To enumerate difunctional correspondences, let us try to get a better representation of how they are "arranged." According to Riguet's theorem (5.1.4), the domain of departure is divided into non-intersecting sets (classes) $\mathbb{A}=\{\mathcal{A}_1..\mathcal{A}_k\}$ by the equivalence $(a_1\sim a_2) \Leftrightarrow R(a_1)=R(a_2)$. (This is really the equivalence: $R(a_1)=R(a_1)$; $R(a_1)=R(a_2)\Leftrightarrow R(a_2)=R(a_1)$;$R(a_1)=R(a_2)\ \&\ R(a_2)=R(a_3)\Rightarrow R(a_1)=R(a_3)$).
5.3.4. However, according to the complementation (5.1.5), the domain of arrival is also divided into non-intersecting sets (classes) $\mathbb{B}=\{\mathcal{B}_1..\mathcal{B}_h\}$ by the equivalence $(b1\sim b2\ ) \Leftrightarrow {}^{R-1}(b1)={}^{R-1}(b2)$ (equivalence is determined in the same way).
5.3.5. Lemma. $((\forall\ a_1,a_2): (a_1\sim a_2))$, $((\forall\ b_1,b_2): (b_1\sim b_2))$ $[(a_1\,R\,b_1) \Leftrightarrow (a_2\,R\,b_2)]$.

According to 5.3.3, $\mathbf{R}(\mathbf{a_1}) = \mathbf{R}(\mathbf{a_2})$**.** Therefore, $(a_1 R b_1) \Rightarrow (a_2 R b_1)$. However, according to 5.3.4, $\boldsymbol{R^{-1}}(\boldsymbol{b_1}) = \boldsymbol{R^{-1}}(\boldsymbol{b_2})$. Therefore, $(a_2 R b_1) \Rightarrow (a_2 R b_2)$.

5.3.6. Let us consider the correspondence between these classes $<\boldsymbol{\mathcal{R}},\mathbb{A},\mathbb{B}>$. Let us assume $(\boldsymbol{\mathcal{A}}\ \boldsymbol{\mathcal{R}}\ \boldsymbol{\mathcal{B}}) \Leftrightarrow (\exists a \in \boldsymbol{\mathcal{A}}, \exists b \in \boldsymbol{\mathcal{B}})\ [aRb]$. (From the lemma 5.3.5, it follows that $(\boldsymbol{\mathcal{A}}\ \boldsymbol{\mathcal{R}}\ \boldsymbol{\mathcal{B}}) \Rightarrow (\forall a \in \boldsymbol{\mathcal{A}}, \forall b \in \boldsymbol{\mathcal{B}})\ [aRb]$). Let us name this correspondence the correspondence generated by the given difunction. Let us give a rigorous definition.

5.3.7. Let us assume that there is a difunctional correspondence $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$**.** Let us name this correspondence the correspondence generated by the given difunction, if

1) The separation $\mathbb{A}$ of the set A is conjugated with the equivalence $(a_1 \sim a_2) \Leftrightarrow R(a_1)=R(a_2)$
2) The separation $\mathbb{B}$ of the set $\boldsymbol{B}$ is conjugated with the equivalence $(b_1 \sim b_2) \Leftrightarrow R(b_1)=R(b_2)$
3) The graph $\boldsymbol{\mathcal{R}}$ is defined by $(\boldsymbol{\mathcal{A}}\ \boldsymbol{\mathcal{R}}\ \boldsymbol{\mathcal{B}}) \Leftrightarrow (\exists a \in \boldsymbol{\mathcal{A}}, \exists b \in \boldsymbol{\mathcal{B}})\ [aRb]$

In this case we will name the difunction $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$ the generating difunction.

5.3.8. Lemma. Each difunction uniquely determines the generated correspondence. Obviously, the separations of the range of definition and the domain of arrival and the correspondence between them are uniquely determined by the graph of the difunction.

5.3.9. Theorem. The correspondence $<\boldsymbol{\mathcal{R}},\mathbb{A},\mathbb{B}>$ generated by any difunction $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$, is functional, injective, total, and pre-surjective.

Let us suppose that $<\boldsymbol{\mathcal{R}},\mathbb{A},\mathbb{B}>$ is not functional.

$(\exists\ \boldsymbol{\mathcal{A}} \in \mathbb{A}, \exists \boldsymbol{\mathcal{B}}_1, \boldsymbol{\mathcal{B}}_2 \in \mathbb{B}, \boldsymbol{\mathcal{B}}_1 \neq \boldsymbol{\mathcal{B}}_2)\ [(\boldsymbol{\mathcal{A}}\ \boldsymbol{\mathcal{R}}\ \boldsymbol{\mathcal{B}}_1) \& (\boldsymbol{\mathcal{A}}\ \boldsymbol{\mathcal{R}}\ \boldsymbol{\mathcal{B}}_2)] \Rightarrow (\exists a \in \boldsymbol{\mathcal{A}}, \exists b_1 \in \boldsymbol{\mathcal{B}}_1, \exists b_2 \in \boldsymbol{\mathcal{B}}_2)\ [(aRb_1) \&\ [(aRb_2)]$. However, according to 5.1.5, $\boldsymbol{R^{-1}}(\boldsymbol{b_1}) \cap \boldsymbol{R^{-1}}(\boldsymbol{b_2}) \neq \emptyset \Rightarrow \boldsymbol{\mathcal{B}}_1 = \boldsymbol{\mathcal{B}}_2$. The contradiction is obtained.

Let us suppose that $<\boldsymbol{\mathcal{R}},\mathbb{A},\mathbb{B}>$ is not injective.

$(\exists\ \boldsymbol{\mathcal{A}}_1, \boldsymbol{\mathcal{A}}_2 \in \mathbb{A}, \boldsymbol{\mathcal{A}}_1 \neq \boldsymbol{\mathcal{A}}_2, \exists\ \boldsymbol{\mathcal{B}} \in \mathbb{B})\ [(\boldsymbol{\mathcal{A}}_1 \boldsymbol{\mathcal{R}} \boldsymbol{\mathcal{B}}) \& (\boldsymbol{\mathcal{A}}_2 \boldsymbol{\mathcal{R}} \boldsymbol{\mathcal{B}})] \Rightarrow (\exists a_1 \in \boldsymbol{\mathcal{A}}_1, \exists a_2 \in \boldsymbol{\mathcal{A}}_2, \exists b \in \boldsymbol{\mathcal{B}})\ [(a_1Rb) \&\ [(a_2Rb)]$. However, according to 5.1.4, $\boldsymbol{R}\ \ (\boldsymbol{a_1}) \cap \boldsymbol{R}\ \ (\boldsymbol{a_2}) \neq \emptyset \Rightarrow \boldsymbol{\mathcal{A}}_1 = \boldsymbol{\mathcal{A}}_2$. The contradiction is obtained.

Let us suppose that $<\boldsymbol{\mathcal{R}},\mathbb{A},\mathbb{B}>$ is not total.

Then $(\exists \boldsymbol{\mathcal{A}}_1, \boldsymbol{\mathcal{A}}_2 \in \mathbb{A}; \boldsymbol{\mathcal{A}}_1 \neq \boldsymbol{\mathcal{A}}_2)\ [\boldsymbol{\mathcal{R}}(\boldsymbol{\mathcal{A}}_1) = \emptyset\ \&\ \boldsymbol{\mathcal{R}}(\boldsymbol{\mathcal{A}}_2) = \emptyset]$. However, in this case $(a_1 \in \boldsymbol{\mathcal{A}}_1)\ [R(a_1) = \emptyset]\ \ (a_2 \in \boldsymbol{\mathcal{A}}_2)\ [R(a_2) = \emptyset]\ \ \Rightarrow R(a_1) = R(a_2) \Rightarrow (a_1 \sim a_2) \Rightarrow \boldsymbol{\mathcal{A}}_1 = \boldsymbol{\mathcal{A}}_2$. The contradiction is obtained.

Let us suppose that $<\boldsymbol{\mathcal{R}},\mathbb{A},\mathbb{B}>$ is not pre-surjective. Then $(\exists \boldsymbol{\mathcal{B}}_1, \boldsymbol{\mathcal{B}}_2 \in \mathbb{B}\ ; \boldsymbol{\mathcal{B}}_1 \neq \boldsymbol{\mathcal{B}}_2)\ [\boldsymbol{\mathcal{R}}^{-1}(\boldsymbol{\mathcal{B}}_1) = \emptyset\ \&\ \boldsymbol{\mathcal{R}}^{-1}(\boldsymbol{\mathcal{B}}_2) = \emptyset]$.

However, in this case $(b_1 \in \boldsymbol{\mathcal{B}}_1)\ [R^{-1}(b_1) = \emptyset]\ \ (b_2 \in \boldsymbol{\mathcal{B}}_2)\ [R^{-1}(b_2) = \emptyset]\ \ \Rightarrow R^{-1}(b_1) = R^{-1}(b_2) \Rightarrow (b_1 \sim b_2) \Rightarrow \boldsymbol{\mathcal{B}}_1 = \boldsymbol{\mathcal{B}}_2$. The contradiction is obtained.

This theorem demonstrates that difunctional correspondences are "predecessors" of functional injection correspondences, and maybe it would be more correct to call them "pre-functional."

5.3.10. Lemma. Different difunctions have different generated correspondences. Let us prove by contradiction.

$V_1 = < \boldsymbol{R_1}, \boldsymbol{A}, \boldsymbol{B} >$ and $V_2 = < \boldsymbol{R_2}, \boldsymbol{A}, \boldsymbol{B} >, \boldsymbol{R_1} \neq \boldsymbol{R_2}$ have the jointly generated correspondence $<\boldsymbol{\mathcal{R}},\mathbb{A},\mathbb{B}>$. $[\exists\, (aR_1b) \,\&\, <a,b> \notin R_2]$. Let us assume that $a \in \boldsymbol{\mathcal{A}}$, $b \in \boldsymbol{\mathcal{B}}$ but then by the definition $(aR_1b) \Rightarrow (\boldsymbol{\mathcal{A}}\, \boldsymbol{\mathcal{R}}\, \boldsymbol{\mathcal{B}}) \Rightarrow (aR_2b)$

5.3.11. Let us denote a set of all functional, injective, pre-total, and pre-surjective correspondences between classes of all possible separations of sets *A,B* by $\mathbb{W}$. Let us also introduce notations for some subsets of $\mathbb{W}$. $\mathbb{W}^C$ is a subset of $\mathbb{W}$ which correspondences have the additional property of complete definability. $\mathbb{W}^S$ is a subset of $\mathbb{W}$ which correspondences have the additional property of surjectivity. $\mathbb{W}^B$ is a subset of $\mathbb{W}$ which correspondences have additional properties of complete definability and surjectivity, i.e. bijective correspondences.

5.4. Theorem on generating difunction

Let us assume that there is the correspondence $<\boldsymbol{\mathcal{R}},\mathbb{A},\mathbb{B}> \in \mathbb{W}$. Then the correspondence $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$ with the graph determined as follows

$$\boldsymbol{R} = \bigcup_{<\boldsymbol{\mathcal{A}},\boldsymbol{\mathcal{B}}> \in \boldsymbol{\mathcal{R}}} \boldsymbol{\mathcal{A}} \times \boldsymbol{\mathcal{B}},$$

is the generating difunction for $<\boldsymbol{\mathcal{R}},\mathbb{A},\mathbb{B}>$.

5.4.1. Let us prove the difunctionality of the built correspondence.

Let us assume that $\boldsymbol{R}(\boldsymbol{a_1}) \cap \boldsymbol{R}(\boldsymbol{a_2}) \neq \emptyset$.

$$\boldsymbol{R}(\boldsymbol{a_1}) \cap \boldsymbol{R}(\boldsymbol{a_2}) \neq \emptyset \;\Rightarrow (\exists \boldsymbol{b} \in \boldsymbol{B})[\, \boldsymbol{a_1 R b} \,\& \boldsymbol{a_2 R b}]$$

According to 3.4.2 $(\exists\, \boldsymbol{\mathcal{A}_1}, \boldsymbol{\mathcal{A}_2} \in \mathbb{A})\, [(\boldsymbol{a_1} \in \boldsymbol{\mathcal{A}_1}), (\boldsymbol{a_2} \in \boldsymbol{\mathcal{A}_2})]$.

By construction $\boldsymbol{R}$

$$\boldsymbol{a_1 R b} \,\& \boldsymbol{a_2 R b} \;\Rightarrow (\exists\, \boldsymbol{\mathcal{B}} \in \mathbb{B})[(\boldsymbol{\mathcal{A}_1 \mathcal{R}\, \mathcal{B}}) \,\&\, (\boldsymbol{\mathcal{A}_2 \mathcal{R}\, \mathcal{B}})]$$

But by condition $<\boldsymbol{\mathcal{R}},\mathbb{A},\mathbb{B}>$ is the injection. $\boldsymbol{\mathcal{A}_1} = \boldsymbol{\mathcal{A}_2}$.

$\boldsymbol{R}(\boldsymbol{a_1}) \cap \boldsymbol{R}(\boldsymbol{a_2}) \neq \emptyset \Rightarrow (\exists\, \boldsymbol{\mathcal{A}} \in \mathbb{A})\, [(\boldsymbol{a_1}, \boldsymbol{a_2} \in \boldsymbol{\mathcal{A}})] \Rightarrow \boldsymbol{R}(\boldsymbol{a_1}) = \boldsymbol{\mathcal{R}}(\boldsymbol{\mathcal{A}}) = \boldsymbol{R}(\boldsymbol{a_2})$.

Thus, we obtain

$$\boldsymbol{R}(\boldsymbol{a_1}) \cap \boldsymbol{R}(\boldsymbol{a_2}) \neq \emptyset \Rightarrow \; \boldsymbol{R}(\boldsymbol{a_1}) = \boldsymbol{R}(\boldsymbol{a_2})$$

According to Riguet's theorem (5.1.4), the correspondence $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$ is difunctional.

5.4.2. Let us prove that the built correspondence is the generating for the given one.

5.4.2.1. Let us prove that the axiom 1 is performed, i.e. that the separation $\mathbb{A}$ of the set A is conjugated with the equivalence $(a_1 \sim a_2) \Leftrightarrow R(a_1)=R(a_2)$.

a) Directly: $(a_1 \sim a_2) \;\Rightarrow R(a_1)=R(a_2)$.

$$(\forall \boldsymbol{\mathcal{A}} \in \mathbb{A})(\, \boldsymbol{a_1}, \boldsymbol{a_2} \in \boldsymbol{\mathcal{A}})[\boldsymbol{R}(\boldsymbol{a_1}) = \boldsymbol{R}(\boldsymbol{a_2})]$$

If $\boldsymbol{R}(\boldsymbol{a_1}) = \boldsymbol{R}(\boldsymbol{a_2}) = \emptyset$, then it is performed in a trivial way. Let us assume that now $\boldsymbol{R}(\boldsymbol{a_1}) \neq \emptyset \Rightarrow (\exists\, \boldsymbol{\mathcal{B}} \in \mathbb{B})\; [\boldsymbol{\mathcal{A}}\, \boldsymbol{\mathcal{R}}\, \boldsymbol{\mathcal{B}})]$. By construction $\boldsymbol{R}(\boldsymbol{a_1}) = \boldsymbol{\mathcal{B}} \,\&\, \boldsymbol{R}(\boldsymbol{a_2}) = \boldsymbol{\mathcal{B}} \Rightarrow \boldsymbol{R}(\boldsymbol{a_1}) = \boldsymbol{R}(\boldsymbol{a_2})$.

b) In reverse: $R(a_1)=R(a_2) \Rightarrow (a_1 \sim a_2)$, which amounts to $(a_1 \nsim a_2) \Rightarrow \; R(a_1) \neq R(a_2)$

$$(\forall \boldsymbol{a_1} \in \mathcal{A}_1, \forall \boldsymbol{a_2} \in \mathcal{A}_2, \mathcal{A}_1 \neq \mathcal{A}_2)[\boldsymbol{R}(\boldsymbol{a_1}) \neq \boldsymbol{R}(\boldsymbol{a_2})]$$

There can be no case $\boldsymbol{R}(\boldsymbol{a_1}) = \emptyset \,\&\, \boldsymbol{R}(\boldsymbol{a_2}) = \emptyset$**,** because then $\mathcal{R}(\mathcal{A}_1) = \emptyset \;\&\; \mathcal{R}(\mathcal{A}_2) = \emptyset, \mathcal{A}_1 \neq \mathcal{A}_2$**,** which contradicts the pre-complete definability <$\mathcal{R}$,$\mathbb{A}$,$\mathbb{B}$>. If only one of $\boldsymbol{R}(\boldsymbol{a_1})$or $\boldsymbol{R}(\boldsymbol{a_2})$ is empty, then $\boldsymbol{R}(\boldsymbol{a_1}) \neq \boldsymbol{R}(\boldsymbol{a_2})$ is performed in a trivial way.

In other cases $(\exists\, \mathcal{B}_1, \mathcal{B}_2 \in \mathbb{B})\ [(\mathcal{A}_1\mathcal{R}\mathcal{B}_1) \,\&\, (\mathcal{A}_2\mathcal{R}\mathcal{B}_2$, owing to injectivity <$\mathcal{R}$,$\mathbb{A}$,$\mathbb{B}$>. $\mathcal{A}_1 \neq \mathcal{A}_2 \Rightarrow \mathcal{B}_1 \neq \mathcal{B}_2$. By construction $(\boldsymbol{R}(\boldsymbol{a_1}) = \mathcal{B}_1) \,\&\, (\boldsymbol{R}(\boldsymbol{a_2}) = \mathcal{B}_2)$ but $(\mathcal{B}_1 \neq \mathcal{B}_2) \Rightarrow \boldsymbol{R}(\boldsymbol{a_1}) \neq \boldsymbol{R}(\boldsymbol{a_2})$.

5.4.2.2. Let us prove that the axiom 2 is performed for the generated correspondence, i.e. that the separation of $\mathbb{B}$ of the set $\boldsymbol{B}$ is conjugated with the equivalence
$(b_1 \sim b_2) \Leftrightarrow R^{-1}(b_1)=R^{-1}(b_2)$.

a) Directly: $(b_1 \sim b_2) \Rightarrow R^{-1}(b_1)=R^{-1}(b_2)$.

$$(\forall \mathcal{B} \in \mathbb{A})(\,\boldsymbol{b_1}, \boldsymbol{b_2} \in \mathcal{B})\left[\boldsymbol{R}^{-1}(\boldsymbol{b_1}) = \boldsymbol{R}^{-1}(\boldsymbol{b_2})\right]$$

If $\boldsymbol{R}^{-1}(\boldsymbol{b_1}) = \boldsymbol{R}^{-1}(\boldsymbol{b_2}) = \emptyset$, then it is performed in a trivial way. Let now be $\boldsymbol{R}^{-1}(\boldsymbol{b_1}) \neq \emptyset \Rightarrow (\exists\, \mathcal{A} \in \mathbb{A})[\mathcal{A}\,\mathcal{R}\,\mathcal{B})]$. By construction $\boldsymbol{R}^{-1}(\boldsymbol{b_1}) = \mathcal{A} \;\&\; \boldsymbol{R}^{-1}(\boldsymbol{b_2}) = \mathcal{A} \Rightarrow \boldsymbol{R}^{-1}(\boldsymbol{b_1}) = \boldsymbol{R}^{-1}(\boldsymbol{b_2})$.

b) In reverse: $R^{-1}(b_1)=R^{-1}(b_2) \Rightarrow (b_1 \sim b_2)$, which amounts to $(b_1 \nsim b_2) \Rightarrow R^{-1}(b_1) \neq R^{-1}(b_2)$

$$(\forall \boldsymbol{b_1} \in \mathcal{B}_1, \forall \boldsymbol{b_2} \in \mathcal{B}_2, \mathcal{B}_1 \neq \mathcal{B}_2)[\boldsymbol{R}^{-1}(\boldsymbol{b_1}) = \boldsymbol{R}^{-1}(\boldsymbol{b_2})]$$

There can be no case $\boldsymbol{R}^{-1}(\boldsymbol{b_1}) = \emptyset \,\&\, \boldsymbol{R}^{-1}(\boldsymbol{b_2}) = \emptyset$ because then $\mathcal{R}^{-1}(\mathcal{B}_1) = \emptyset \;\&\; \mathcal{R}^{-1}(\mathcal{B}_2) = \emptyset, \mathcal{B}_1 \neq \mathcal{B}_2$**,** which contradicts the pre-surjectivity <$\mathcal{R}$,$\mathbb{A}$,$\mathbb{B}$>. If only one of $\boldsymbol{R}^{-1}(\boldsymbol{b_1})$ or $\boldsymbol{R}^{-1}(\boldsymbol{b_2})$ is empty, then $\boldsymbol{R}^{-1}(\boldsymbol{b_1}) \neq \boldsymbol{R}^{-1}(\boldsymbol{b_2})$ is performed in a trivial way.

In other cases $(\exists\, \mathcal{A}_1, \mathcal{A}_2 \in \mathbb{A})\ [(\mathcal{A}_1\mathcal{R}\mathcal{B}_1) \,\&\, (\mathcal{A}_2\mathcal{R}\mathcal{B}_2]$, owing to functionality <$\mathcal{R}$,$\mathbb{A}$,$\mathbb{B}$>, $\mathcal{B}_1 \neq \mathcal{B}_2 \Rightarrow \mathcal{A}_1 \neq \mathcal{A}_2$. By construction $(\boldsymbol{R}^{-1}(\boldsymbol{b_1}) = \mathcal{A}_1) \,\&\, (\boldsymbol{R}^{-1}(\boldsymbol{b_2}) = \mathcal{A}_2)$ but $(\mathcal{A}_1 \neq \mathcal{A}_2) \Rightarrow \boldsymbol{R}^{-1}(\boldsymbol{b_1}) \neq \boldsymbol{R}^{-1}(\boldsymbol{b_2})$.

5.4.2.3. Let us prove that axiom 3 for the generated correspondence is performed, i.e. $(\mathcal{A}\,\mathcal{R}\,\mathcal{B}) \Leftrightarrow (\exists a \in \mathcal{A}, \exists b \in \mathcal{B})$ [aRb].
a) Directly: $(\mathcal{A}\,\mathcal{R}\,\mathcal{B}) \Rightarrow (\exists a \in \mathcal{A}, \exists b \in \mathcal{B})$ [aRb]. Immediate from construction $(\mathcal{A}\,\mathcal{R}\,\mathcal{B}) \Rightarrow (\forall \in \mathcal{A}, \forall b \in \mathcal{B})$ [aRb].
b) In reverse: $(\exists a \in \mathcal{A}, \exists b \in \mathcal{B})$ [aRb] $\Rightarrow (\mathcal{A}\,\mathcal{R}\,\mathcal{B})$ – by construction and owing to uniqueness 3.4.2 of the class of separation to which each element belongs.

5.5. The first theorem on bijection of difunctions

Theorem. The correspondence <$\mathfrak{R}$,W^D,$\mathbb{W}$> which assigns to each difunction from the domain of departure the correspondence from the domain of arrival generated by it, is a bijection.

5.5.1. It is shown in 5.3.8 that each difunction generates exactly one correspondence. According to the theorem 5.3.9, all of such correspondences belong to $\mathbb{W}$. Then $(\forall\, w \in W^{\mathrm{D}})[|\mathfrak{R}(\mathrm{w}) = 1|]$.

5.5.2. The theorem 5.4 shows that for any correspondence from $\mathbb{W}$ there is at least one generating difunction from W^D. The lemma 5.3.10 shows that there is no more than one of them. Then $(\forall\, \mathbb{w} \in \mathbb{W})[|\mathfrak{R}^{-1}(\mathbb{w}) = 1|]$.

5.5.3. According to lemma 3.2.6, <$\mathfrak{R}$,W^D,$\mathbb{W}$> is the bijection.

5.6. The second theorem on bijection of difunctions

Theorem. The correspondence <$\mathfrak{R}$,W^{DC},$\mathbb{W}^C$> which assigns to each difunction from the domain of departure the correspondence from the domain of arrival generated by it, is a bijection.

5.6.1. Lemma. The correspondence <$\boldsymbol{\mathcal{R}}$,$\mathbb{A}$,$\mathbb{B}$> generated by some difunction $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$ is total if and only if when $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$ is total.

5.6.1.1. Let us make direct proof by contradiction.

Let us assume that <$\boldsymbol{\mathcal{R}}$,$\mathbb{A}$,$\mathbb{B}$> is not total, and $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$ is total. $(\exists\, \boldsymbol{\mathcal{A}} \in \mathbb{A})\ [\boldsymbol{\mathcal{R}}(\boldsymbol{\mathcal{A}})=\emptyset]$. According to 3.4.3, $(\exists\ \mathrm{a} \in \boldsymbol{\mathcal{A}})[\mathrm{a} \in \mathrm{A}]$, since $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$ is total $(\exists \mathbf{b} \in \mathbf{B})[\mathbf{aRb}]$. According to 3.4.2, $(\exists \boldsymbol{\mathcal{B}} \in \mathbb{B})[\mathrm{b} \in \boldsymbol{\mathcal{B}}]$. But then by the definition of the generated correspondence $\boldsymbol{\mathcal{A}}\ \boldsymbol{\mathcal{R}}\ \boldsymbol{\mathcal{B}} \Rightarrow \boldsymbol{\mathcal{R}}(\boldsymbol{\mathcal{A}}) \neq \emptyset$ **.** The contradiction is obtained.

5.6.1.2. Let us make proof by contradiction.

Let us assume that <$\boldsymbol{\mathcal{R}}$,$\mathbb{A}$,$\mathbb{B}$> is the total correspondence. Owing to lemma 3.4.2, $(\forall \boldsymbol{a} \in \boldsymbol{A})(\exists \boldsymbol{\mathcal{A}} \in \mathbb{A})[\boldsymbol{a} \in \boldsymbol{\mathcal{A}}]$. $(\exists \boldsymbol{\mathcal{B}} \in \mathbb{B})\ [\boldsymbol{\mathcal{A}}\ \boldsymbol{\mathcal{R}}\ \boldsymbol{\mathcal{B}}] \Rightarrow (\exists \mathrm{b} \in \boldsymbol{\mathcal{B}})[\mathrm{aRb}]$ (by construction of R).

5.6.2. Owing to 3.3 The correspondence <$\mathfrak{R}$,W^{DC},$\mathbb{W}^C$> is the bijection.

5.7. The third theorem on bijection of difunctions

Theorem. The correspondence <$\mathfrak{R}$,W^{DS},$\mathbb{W}^S$> which assigns to each difunction from the domain of departure the correspondence from the domain of departure generated by it, is a bijection.

5.7.1. Lemma. The correspondence <$\boldsymbol{\mathcal{R}}$,$\mathbb{A}$,$\mathbb{B}$> generated by some difunction $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$ is surjective if and only if when $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$ is surjective.

5.7.1.1. Let us make direct proof by contradiction.

Let us assume that <$\boldsymbol{\mathcal{R}}$,$\mathbb{A}$,$\mathbb{B}$> is not surjective, and $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$ is surjective. $(\exists \boldsymbol{\mathcal{B}} \in \mathbb{B})\ [\boldsymbol{\mathcal{R}}^{-1}(\boldsymbol{\mathcal{B}})=\emptyset]$. According to 3.4.3, $(\exists \mathbf{b} \in \mathbb{B})[\mathrm{b} \in \mathrm{B}]$, since $< \boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B} >$ is surjective $(\exists \mathbf{a} \in \mathbf{A})[\mathbf{aRb}]$. According to 3.4.2, $(\exists \boldsymbol{\mathcal{A}} \in \mathbb{A})[\mathrm{a} \in \boldsymbol{\mathcal{A}}]$. But then by the definition of the generated correspondence $(\boldsymbol{\mathcal{A}}\ \boldsymbol{\mathcal{R}}\ \boldsymbol{\mathcal{B}}) \Rightarrow \boldsymbol{\mathcal{R}}^{-1}(\boldsymbol{\mathcal{B}}) \neq \emptyset$. The contradiction is obtained.

5.7.1.2. Let us make proof by contradiction.

Owing to lemma 3.4.2, $(\forall \boldsymbol{b} \in \boldsymbol{B})(\exists \boldsymbol{\mathcal{B}} \in \mathbb{B})[\mathrm{b} \in \boldsymbol{\mathcal{B}})]$. Since $\mathbb{W}^{S}$ is the set of surjective correspondence, then $(\exists \boldsymbol{\mathcal{A}} \in \mathbb{A})[\boldsymbol{\mathcal{A}}\, \boldsymbol{\mathcal{R}}\, \boldsymbol{\mathcal{B}}] \Rightarrow (\exists \boldsymbol{a} \in \boldsymbol{\mathcal{A}}) \Rightarrow [\mathrm{aRb}]$ (by construction of R).

5.7.2. Owing to 3.3, the correspondence $<\boldsymbol{\mathfrak{R}}, W^{DS}, \mathbb{W}^{S}>$ is the bijection.

5.8. The fourth theorem on bijection of difunctions

Theorem. The correspondence $<\boldsymbol{\mathfrak{R}}, W^{DCS}, \mathbb{W}^{B}>$ which assigns to each difunction from the domain of departure the correspondence from the domain of arrival generated by it, is a bijection.

5.8.1. Lemma. The correspondence $<\boldsymbol{\mathcal{R}}, \mathbb{A}, \mathbb{B}>$ generated by some difunction $<\boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B}>$ is the bijection if and only if when $<\boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B}>$ is total and surjective.

5.8.1.1. Let us make a direct proof. Let us assume that $<\boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B}> \in W^{DCS}$. Then owing to 5.6.1, 5.7.1, $<\boldsymbol{\mathcal{R}}, \mathbb{A}, \mathbb{B}>$ is the total and surjective correspondence. But owing to 5.3.8, it is injective and functional, i.e. is the bijection.

5.8.1.2. Let us make proof by contradiction. $<\boldsymbol{\mathcal{R}}, \mathbb{A}, \mathbb{B}>$ is the bijection, and therefore, is the total and surjective correspondence. Then owing to 5.6.1 and 5.7.1, $<\boldsymbol{R}, \boldsymbol{A}, \boldsymbol{B}>$ has properties of complete definability and surjectivity.

5.8.2. Owing to 3.3 The correspondence $<\boldsymbol{\mathfrak{R}}, W^{DCS}, \mathbb{W}^{B}>$ is the bijection.

5.9. Enumeration of pre-surjective and total correspondences

Theorems 5.5-5.8 demonstrate that pre-surjective and total injections are related to difunctions. Let us enumerate them – it will be necessary for enumerating the difunctions later.

5.9.1. Lemma. The number of functional, injective, total, and pre-surjective correspondences between the n-set and m-set is equal to

$$\boldsymbol{\delta_n^m * n! + \delta_{n+1}^m * (n+1)!}$$

Let us divide the set of these correspondences into two non-intersecting classes –surjective and non-surjective. Then the surjective correspondences by condition are bijections, and their number is $\boldsymbol{\delta_n^m * n!}$.

Non-surjective correspondences contain by condition exactly one element that does not fall into the range of values. The number of ways to select this element is m. The other m-1 elements must be in the bijection with the domain of departure. Their number is $\boldsymbol{\delta_n^{m-1} * (m-1)!}$. By the multiplication rule $\boldsymbol{\delta_n^{m-1} * (m-1)! * m = \delta_n^{m-1} * m!}$. Obviously, $\boldsymbol{\delta_n^{m-1} = \delta_{n+1}^m}$. Besides, $\boldsymbol{\delta_{n+1}^m = 1 \Leftrightarrow m = n+1}$.

We obtain $\boldsymbol{\delta_{n+1}^m * (n+1)!}$ of non-surjective correspondences. Finally, by the addition rule (2.3), we obtain the result:

$$\boldsymbol{\delta_n^m * n! + \delta_{n+1}^m * (n+1)!}$$

5.9.2. Lemma. The number of functional, injective, surjective, and pre-total correspondences between the n-set and m-set is equal to

$$\boldsymbol{\delta_n^m * n! + \delta_{n-1}^m * n!}$$

Let us divide the set of these correspondences into two non-intersecting classes – total and non-total. Then the total correspondences by condition are bijections, and their number is $\boldsymbol{\delta_n^m * n!}$.

Non-total correspondences contain by condition exactly one element that does not fall into the range of definition. The number of ways to select this element is n. The other n-1 elements must be in the bijection with the domain of arrival. Their number is $\boldsymbol{\delta_{n-1}^m * (n-1)!}$. Finally, by the multiplication rule (2.3), we obtain the result:

$$\boldsymbol{\delta_n^m * n! + \delta_{n-1}^m * n!}$$

5.9.3. Lemma. The number of functional, injective, pre-surjective, and pre-total correspondences between the n-set and m-set is equal to

$$\boldsymbol{(\delta_{n-1}^m * n! + \delta_n^m * (n+1)! + \delta_{n+1}^m * (n+1)!}$$

Let us divide the set of these correspondences into four non-intersecting classes – total non-surjective ($\boldsymbol{\delta_{n-1}^m * n!}$), surjective non-total ($\boldsymbol{\delta_{n+1}^m * (n+1)!}$), bijective ($\boldsymbol{\delta_n^m * n!}$), and non-surjective non-total. Only the number of non-surjective non-total correspondences remains to be calculated. By the multiplication rule, we obtain $\boldsymbol{n * m * \delta_{n-1}^{m-1} * (n-1)!}$, where n is the number of ways to select the element, which does not fall into the range of definition, m is the number of ways to select the element, which does not fall into the range of values, $\boldsymbol{\delta_{n-1}^{m-1} * (n-1)!}$ is the number of ways to determine the bijection between the elements remaining in the range of definition and the range of values.

$$\boldsymbol{n * m * \delta_{n-1}^{m-1} * (n-1)! = \delta_n^m * n * m * (n-1)! = \delta_n^m * n * n * (n-1)!}$$
$$\boldsymbol{= \delta_n^m * n * n!}$$

By the addition rule, we obtain the result:

$$\boldsymbol{\delta_{n-1}^m * n + \delta_{n+1}^m * (n+1)! + \delta_n^m * n! + \delta_n^m * n * n!}$$
$$\boldsymbol{= \delta_{n-1}^m * n + \delta_{n+1}^m * (n+1)! + \delta_n^m * (n+1) * n!}$$
$$\boldsymbol{= \delta_{n-1}^m * n! + \delta_n^m * (n+1)! + \delta_{n+1}^m * (n+1)!}$$

5.10. Enumeration of difunctions

Theorems 5.5-5.8 allow for the enumeration of difunctional correspondences to enumerate correspondences on the separations of the domain of departure and domain of arrival with definite properties (2.1.1). In doing so, we use the Stirling transform.

5.10.1. Let us divide each of the sets $\mathbb{W}$, $\mathbb{W}^B$, $\mathbb{W}^S$, $\mathbb{W}^C$ into non-intersecting classes for the pair of the cardinal of the domain of departure $|\mathbb{A}|$ and cardinal of the domain of arrival $|\mathbb{B}|$. According to (3.4.5), $1 \leq |\mathbb{A}| \leq n$, $1 \leq |\mathbb{B}| \leq m$. The number of correspondences from a given set for a fixed pair $\boldsymbol{<i, j>}$ we will denote by $w(i, j)$.

Applying the Stirling transform (2.7) twice, we get the formula:

$$\sum_{i=1}^{n}\left(\boldsymbol{S_n^i} * \sum_{j=1}^{m}\boldsymbol{S_m^j} * w(i, j)\right)$$

where $w(i, j)$ is the number of correspondences between the i-set and j-set having the predefined set of properties.

5.10.2. Theorem. $|W^{DCS}| = \sum_{i=1}^{n} S_n^i * S_m^i * i!$ **.**

Let us make use of (5.8) and (2.1.1) and calculate $|\mathbb{W}^B|$.

For reasons given in 5.10.1,

$|\mathbb{W}^B| = \sum_{i=1}^{n}(S_n^i * \sum_{j=1}^{m} S_m^j * w(i,j))$, here $w(i,j)$ is the number of bijections between the i-set and j-set. For reasons given in 4.2.4.1, $w(i,j) = \delta_i^j * i!$

$$\sum_{i=1}^{n}\left(S_n^i * \sum_{j=1}^{m} S_m^j * \delta_i^j * i!\right) = \sum_{i=1}^{n}\left(S_n^i * i! * \sum_{j=1}^{m} \delta_i^j * S_m^j\right).$$

For reasons given in 3.5.2,

$$\sum_{j=1}^{m} \delta_i^j * S_m^j = S_m^i$$

we reduce and obtain:

$$\sum_{i=1}^{n} S_n^i * S_m^i * i!$$

5.10.3. Theorem. $|W^{DC}| = \sum_{i=1}^{n} S_n^i * (S_m^i * i! + S_m^{i+1} * (i+1)!)$**.**

Let us make use of (5.8) and (2.1.1) and calculate $|\mathbb{W}^C|$.

For reasons given in 5.10.1,

$|\mathbb{W}^C| = \sum_{i=1}^{n}(S_n^i * \sum_{j=1}^{m} S_m^j * w(i,j))$, here $w(i,j)$ is the number of F, C, I, and pre-surjective correspondences between the i-set and j-set. For reasons given in 5.9.1, $w(i,j) = \delta_i^j * i! + \delta_{i+1}^j * (i+1)!$ Let us apply to 5.10.1

$$\sum_{i=1}^{n}\left(S_n^i * \sum_{j=1}^{m} S_m^j * (\delta_i^j * i! + \delta_{i+1}^j * (i+1)!)\right) =$$

$$\sum_{i=1}^{n}\left(S_n^i * \left(\sum_{j=1}^{m} S_m^j * \delta_i^j * i! + \sum_{j=1}^{m} S_m^j * \delta_{i+1}^j * (i+1)!\right)\right)$$

Let us use 3.5.2 to simplify:

$$\sum_{j=1}^{m} S_m^j * \delta_i^j * i! = S_m^i * i!; \quad \sum_{j=1}^{m} S_m^j * \delta_{i+1}^j * (i+1)! = S_m^{i+1} * (i+1)!$$

We obtain the result:

$$\sum_{i=1}^{n} S_n^i * (S_m^i * i! + S_m^{i+1} * (i+1)!)$$

5.10.4. Theorem. $|W^{DS}| = \sum_{i=1}^{n} S_n^i * (S_m^{i-1} * i! + S_m^i * i!)$.

Let us make use of (5.7) and (2.1.1) and calculate $|\mathbb{W}^S|$.

For reasons given in 5.10.1, $|\mathbb{W}^C|=\sum_{i=1}^{n}(\boldsymbol{S_n^i} * \sum_{j=1}^{m} \boldsymbol{S_m^j} * w(i,j))$, here $w(i,j)$ is the number of F, I, S, and pre-total correspondences between the i-set and j-set. For reasons given in 5.9.2, $w(i,j) = \boldsymbol{\delta_i^j * i! + \delta_{i-1}^j * i!}$ **.** Let us apply to 5.10.1

$$\sum_{i=1}^{n}\left(\boldsymbol{S_n^i} * \sum_{j=1}^{m} \boldsymbol{S_m^j} * (\boldsymbol{\delta_i^j} * \boldsymbol{i}! + \boldsymbol{\delta_{i-1}^j} * \boldsymbol{i}!)\right) =$$

$$\sum_{i=1}^{n}\left(\boldsymbol{S_n^i} * \left(\sum_{j=1}^{m} \boldsymbol{S_m^j} * \boldsymbol{\delta_i^j} * \boldsymbol{i}! + \sum_{j=1}^{m} \boldsymbol{S_m^j} * \boldsymbol{\delta_{i-1}^j} * \boldsymbol{i}!\right)\right) =$$

$$= \sum_{i=1}^{n} \boldsymbol{S_n^i} * \left(\boldsymbol{S_m^i} * \boldsymbol{i}! + \boldsymbol{S_m^{i-1}} * \boldsymbol{i}!\right)$$

Here, as before, 3.5.2 was used for the simplification.

5.10.5. Theorem.

$|\mathrm{W}^D|= \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}(\boldsymbol{S_n^i} * \left(\boldsymbol{S_m^{i-1}} * \boldsymbol{i}! + \boldsymbol{S_m^i} * (\boldsymbol{i}+\boldsymbol{1})! + \boldsymbol{S_m^{i+1}} * (\boldsymbol{i}+\boldsymbol{1})!\right))$

Let us make use of (5.5) and (2.1.1) and calculate $|\mathbb{W}|$.

For reasons given in 5.10.1, $|\mathbb{W}^C|=\sum_{i=1}^{n}(\boldsymbol{S_n^i} * \sum_{j=1}^{m} \boldsymbol{S_m^j} * w(i,j))$, here $w(i,j)$ is the number of F, I, pre-surjective, and pre-total correspondences between the i-set and j-set. For reasons given in 5.9.3

$\boldsymbol{w(i,j) = \delta_{i-1}^j * i! + \delta_i^j * (i+1)! + \delta_{i+1}^j * (i+1)!}$ **.**

Let us apply to 5.10.1

$$\sum_{i=1}^{n}\left(\boldsymbol{S_n^i} * \sum_{j=1}^{m} \boldsymbol{S_m^j} * \left(\boldsymbol{\delta_{i-1}^j} * \boldsymbol{i}! + \boldsymbol{\delta_i^j} * (\boldsymbol{i}+\boldsymbol{1})! + \boldsymbol{\delta_{i+1}^j} * (\boldsymbol{i}+\boldsymbol{1})!\right)\right) =$$

$$\sum_{i=1}^{n}\left(\boldsymbol{S_n^i} * \left(\sum_{j=1}^{m} \boldsymbol{S_m^j} * \boldsymbol{\delta_{i-1}^j} * \boldsymbol{i}! + \sum_{j=1}^{m} \boldsymbol{S_m^j} * \boldsymbol{\delta_i^j} * (\boldsymbol{i}+\boldsymbol{1})! + \sum_{j=1}^{m} \boldsymbol{S_m^j} * \boldsymbol{\delta_{i+1}^j} * (\boldsymbol{i}+\boldsymbol{1})!\right)\right) =$$

$$\sum_{i=1}^{n}\left(\boldsymbol{S_n^i} * (\boldsymbol{S_m^{i-1}} * \boldsymbol{i}! + \boldsymbol{S_m^i} * (\boldsymbol{i}+\boldsymbol{1})! + \boldsymbol{S_m^{i+1}} * (\boldsymbol{i}+\boldsymbol{1})!)\right)$$

Here, as before, 3.5.2 was used for the simplification.

6. Results

6.1. The result of the work is summarized in the table of Appendix 1. All formulas have been found.

6.2. The enumeration of correspondences with a combination of functionality, complete definability, injectivity, surjectivity, and difunctionality proved to be an easier problem than the enumeration of relations over set where many important problems are not solved, such as enumeration of serial relations.

6.3. To evaluate the correctness of formulas, a program has been developed that calculates the first values of sequences by brute force.

6.4. To check the sequences in OEIS and to add new sequences, a program has been developed that calculates the values by formulas and represents sequences in both tabular and "antidiagonal" form.

6.5. Some of the problems are reformulated classical problems. First of all, this refers to strings 1, 4, 12, 13, 15, and 16.

6.6. Some of the problems are mutually symmetric. Problems 19, 20, and 23-32 completely coincide with problems 3, 4, and 7-16, respectively. The reason is determined in the work.

6.7. For the absolute majority of problems, the sequences found have been already registered in OEIS.

6.7.1. Many of the sequences obtained are formulated in OEIS as enumeration problems of binary m-by-n matrices, and their connection to the correspondences between n-sets and m-sets is evident.

6.7.2. Some of the problems are formulated in OEIS as enumeration problems of correspondences between strings of length n and m. It is very close to our formulation.

6.8. New results (strings 17, 18, and 21) have been gained when enumerating difunctional correspondences. These results have been registered in OEIS with the numbers A265417, A265706, and A265707.

6.9. In all these correspondences, the domains of departure and arrival were considered as labeled sets. In related problems of enumeration of graphs or relations over set also the problems of enumeration with unlabeled sets are solved. Usually, these are more complex problems. In the future, it will be interesting to investigate also our enumerations for unlabeled sets.

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Annex 1

	D	F	C	I	S		Ref	OEIS
1						2^{mn}		
2					+	$(2^n-1)^m$	L_5	A245789
3				+		$(n+1)^m$	L_9	A001597
4				+	+	n^m	L_{13}	A001597

5			+			$(2^m-1)^n$	L_2	A245789
6			+		+	$(2^m-1)^n-\sum_{i=1}^{m-1} C_m^i * L_6(n,i)$		A183109
7			+	+		$S_m^{n+1}*(n+1)!+S_m^n*n!$	L_{10}	A142071
8			+	+	+	$S_m^n*n!$	L_{14}	A019538
9		+				$(m+1)^n$	L_3	A001597
10		+			+	$S_n^{m+1}*(m+1)!+S_n^m*m!$	L_7	A142071
11		+		+		$\sum_{i=0}^{n} C_n^i * C_m^i * i!$		A088699
12		+		+	+	$A_n^m=C_n^m*m!$	L_{15}	A002720
13		+	+			m^n	L_4	A001597
14		+	+		+	$S_n^m*m!$	L_8	A019538
15		+	+	+		$A_m^n=C_m^n*n!$	L_{12}	A002720
16		+	+	+	+	$\delta_n^m*n!$		A000142
17	+					$\sum_{i=1}^{n} S_n^i*(S_m^{i-1}*i!+S_m^i*(i+1)!+S_m^{i+1}*(i+1)!)$		A265417
18	+				+	$\sum_{i=1}^{n} S_n^i*(S_m^{i-1}*i!+S_m^i*i!)$		A265707
19	+			+		$(n+1)^m$	L_3	A001597
20	+			+	+	n^m	L_4	A001597
21	+		+			$\sum_{i=1}^{n} S_n^i*(S_m^i*i!+S_m^{i+1}*(i+1)!)$		A265706
22	+		+		+	$\sum_{i=1}^{n} S_n^i*S_m^i*i!$		A108470
23	+		+	+		$S_m^{n+1}*(n+1)!+S_m^n*n!$	L_7	A142071
24	+		+	+	+	$S_m^n*n!$	L_8	A019538
25	+	+				$(m+1)^n$	L_9	A001597
26	+	+			+	$S_n^{m+1}*(m+1)!+S_n^m*m!$	L_{10}	A142071
27	+	+		+		$\sum_{i=0}^{n} C_n^i * C_m^i * i!$	L_{11}	A142071
28	+	+		+	+	$A_n^m=C_n^m*m!$	L_{12}	A002720
29	+	+	+			m^n	L_{13}	A001597
30	+	+	+		+	$S_n^m*m!$	L_{14}	A019538
31	+	+	+	+		$A_m^n=C_m^n*n!$	L_{15}	A002720
32	+	+	+	+	+	$\delta_n^m*n!$	L_{16}	A000142